Fuzzy Logic, Probability and the strange bond...

PhD Course: Mathematical Logic

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Summary

An overview of the presentation

- Fuzzy Logic Theory
- Probability Theory
- Fuzzy Logic and Probability: the strange bond

Many-Valued Logic

- In **Classical Logic** (i.e. Aristotle's Bivalent Logic) there are only two possible values for any proposition (true and false)
- In **Many-Valued Logic** (also called Multi-Valued Logic) any proposition can assume more than two truth values
- In literature, there are several extensions to the to classical two-valued logic [Gottwald(2009)]:
  * the Three-Valued Logic (e.g. Lukasiewicz's and Kleene's, which accept the values true, false, and unknown)
  * the Finite n-Valued Logic with more than three values
  * the Infinite-Valued Logic (e.g. Fuzzy Logic and Probabilistic Logic)

Fuzzy Logic

- **Classical Logic** only deals with precise information or exact data (i.e. assertions completely true or completely false), while **Fuzzy Logic** is also appropriate for the treatment of imprecise information or approximate data (i.e. vague assertions which can be partially true or partially false)
- NOTE: the reasoning in Fuzzy Logic is approximate and therefore is most similar to the human reasoning with respect to Classical Logic

t-norm and t-conorm

- **triangular norm** (t-norm) **is a binary operation used in the probabilistic metric spaces and in Multi-Valued Logic**, specifically in Fuzzy Logic [Hajek(2010)]
- triangular refers to the fact that in the probabilistic metric spaces t-norms are used to **generalize triangle inequality of ordinary metric spaces**
- t-norms are a generalization of the classical two-valued logical conjunction for the Fuzzy Logics
- A t-norm is a function $T : [0,1] \times [0,1] \rightarrow [0,1]$ which satisfies the following properties:
  * Commutativity: $T(a,b) = T(b,a)$
  * Monotonicity: if $a \leq b$ then $T(a,c) \leq T(b,c)$
  * Associativity: $T(a, T(b,c)) = T(T(a,b),c)$
  * The number 1 acts as identity element: $T(a,1) = a$
- t-conorms (or S-norms) are dual to t-norms under the order-reversing operation which assigns $1-a$ to a on $[0,1]$. Given a t-norm, the complementary conorm is defined by $\perp(a,b) = 1 - T(1-a, 1-b)$
- NOTE: this generalizes De Morgan's laws

Residuum

- A t-norm is continuous if it is continuous as a function in the topology on interval $[0,1] \times [0,1]$
- The continuity of $\&_T$ is translated by the axiom $a \&_T (a \rightarrow b) \equiv a \wedge b$
- **For any left-continuous t-norm T, there is a unique binary operation, the residuum, on** $[0,1]$ such that $a \rightarrow_T b = sup\{c \mid T(a,c) \leq b\}$
- **The residuum** (often called R-implication) **is the implication in the standard semantics of t-norm based fuzzy logics**
- A t-norm and its residuum are connected by the residuation property $a \&_T b \leq c \equiv T(a,b) \leq c \iff a \leq (b \rightarrow_T c)$

Most important continuous t-norm

	$x * y$	$x \Rightarrow y$ for $x \leq y$	$x \Rightarrow y$ for $x > y$	$\neg x$
Łukasiewicz	$\max(0, x + y - 1)$	1	$1 - x + y$	$1 - x$
Gödel	$\min(x, y)$	1	y	$\neg 0 = 1$
product	$x \cdot y$	1	y/x	$\neg x = 0$ for $x > 0$

Figure: Most important t-norms, their residues and negations [Hajek(2010)]

- NOTE: Godel's t-norm is the only t-norm that is idempotent i.e. $T(a, a) = a$
- NOTE: Lukasiewicz's t-norm is the only continuous t-norm whose residuum is continuous and for which the translation of implication $a \rightarrow b$ by $\neg a \&_{\perp} b$ is consistent

Fuzzy Logic Semantic

- Truth values are totally ordered real numbers in $S = [0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ where the order is the order of $[0, 1]$
- **Fuzzy Logic is truth functional**, i.e. the truth value of a compound formula is uniquely determined by the truth values of its components
- Fuzzy Logic has in general two conjunctions:
  - the weak conjunction $\wedge$, $a \wedge b = min\{a, b\}$
  - the **strong conjunction** $\&_T$ which has as corresponding truth degree function a t-norm T

  and the corresponding two disjunctions:
  - the weak disjunction $\vee$, $a \vee b = max\{a, b\}$
  - the **strong disjunction** $\&_\perp$ which has as corresponding truth degree function a t-conorm $\perp$
- the **negation connective**, $\neg_T$, is determined by the truth degree function $\neg_T a = a \rightarrow_T 0$

Propositional Fuzzy Logic

- The most important propositional Fuzzy Logics are [Hajek(2010)]:
  * *Monoidal t-norm-based propositional Fuzzy Logic* (MTL) is an axiomatization of logic where conjunction is defined by a left continuous t-norm, and implication is defined as the corresponding residuum
  * *Basic propositional Fuzzy Logic* (BL) is an extension of MTL logic where conjunction is defined by a continuous t-norm, and implication is also defined as the residuum of the t-norm
  * *Lukasiewicz's Fuzzy Logic* (LFL) is the extension of BL where standard conjunction is the Lukasiewicz's t-norm. It has the axioms of BL plus an axiom of double negation
  * *Godel's Fuzzy Logic* (GFL) is the extension of BL where conjunction is Godel's t-norm. It has the axioms of BL plus an axiom of idempotence of conjunction
  * *Product Fuzzy Logic* (PFL) is the extension of BL where conjunction is product t-norm. It has the axioms of BL plus another axiom for cancellativity of conjunction i.e. for any a, b, c with $c \neq 0$ if $T(a, c) = T(b, c)$ then $a = b$

Approaches to Fuzzy Logic

- There are two approaches to Fuzzy Logic:
  * **traditional** [Hajek(2010)]: a set of possible values is fixed and this allows to define an entailment relation. The logical consequence operator gives the set of logical consequence of a given set of axioms
  * **modern** [Hajek(2010)]: the aim is to defining a deduction apparatus in which approximate reasoning is admitted. The logical consequence operator gives the fuzzy subset of logical consequence of a given fuzzy subset of hypotheses
- NOTE: *Rational Pavelka's Logic* (RPL) is a Fuzzy Logic with evaluated syntax and is a further generalization of Fuzzy Logic. While the traditional kinds of Fuzzy Logic have traditional syntax and many-valued semantics, in RPL is evaluated also syntax. This means that each formula has an evaluation

Predicate Fuzzy Logic

- The predicate Fuzzy Logic is obtained by adding the universal and the existential quantifiers to the propositional Fuzzy Logic. This is similar to the way that predicate logic is created from propositional logic
- **The universal and existential quantifiers** in t-norm Fuzzy Logics are interpreted as the **infimum and supremum** respectively of the truth degrees of the instances of the quantified subformula
- NOTE: the class of all t-norms is very large, and up to now not really well understood. Actually one is able to axiomatize t-norm based systems for some particular classes of t-norms [Hajek(2010)]
- NOTE: the question about t-norm based quantifiers is a recent research problem [Hajek(2010)]

Fuzzy Set Theory

- The notion of **fuzzy subset** was introduced by Zadeh as **a formalization of vagueness**, i.e. a predicate may apply to an object not absolutely, but to a certain degree, and so there may be borderline cases
- t-norms are also used to construct the intersection of fuzzy sets or as a basis for aggregation operators
- NOTE: fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1

Fuzzy Computing I

- Given partial imprecise knowledge express in Fuzzy Logic as IF-THEN rules on some crisp functions of the form IF x is A_i THEN y is B_i [Hajek(2010)]
- The rules can be viewed as set of relations between the values of a (dependent) variable y to values of an (independent) variable x i.e. for each $x = A_i$ there is $y = B_i$ where A_i, B_i may be crisp or fuzzy values
- There are two, in general non-equivalent, ways of interpretation:
  * **Mamdani's formula** as listing of n possibilities

$$MAMD(x, y) \equiv \bigvee_{i=1}^{n} (A_i(x) \&_T B_i(y))$$

  * **conjunction of implications**

$$RULES(x, y) \equiv \bigwedge_{i=1}^{n} (A_i(x) \rightarrow_T B_i(y))$$

- NOTE: both $MAMD$ and $RULES$ define a binary fuzzy relation (given the interpretation of A_i, B_i and truth functions of connectives)

Fuzzy Computing II

- The operator $B(y) \equiv \exists x(A(x) \&_T R(x, y))$ assigns to each fuzzy input set $A(x)$ a corresponding fuzzy output set $B(y)$, where $R(x, y)$ is $MAMD(x, y)$ (most frequent case) or $RULES(x, y)$
- A **fuzzifications** is a procedure that allows to convert a crisp input x_i into some fuzzy input $A(x)$ (x_i is similar to x) while a **defuzzification** is able to convert the fuzzy image $B(y)$ to a crisp output y_i
- NOTE: $A(x)$ and $B(y)$ are generic fuzzy sets: they can also be coincident although this concept might be different from standard set theory [Hajek(2010)]

Fuzzy Logic and Classical Logic

- **Fuzzy Logic rejects** the **law of the excluded middle** $\neg a \vee a$ and the law of contraposition $a \rightarrow b = \neg b \rightarrow \neg a$
- **In Fuzzy Logic** a **generic propositional formula cannot be expressed in Conjunctive Normal Form (CNF) and Disjunctive Normal Form (DNF)**: this is due to the representation of the implication through the residuum. However, some Fuzzy Logics, like Lukasiewicz's Fuzzy Logic, allow the CNF and DNF
- **Fuzzy Logic** can **partially solve the liar paradox** and can **solve the Sorites paradox** [Hajek(2010)]

Introduction to Probability Theory

- **The definition of probability of an event is an hard task** (still in progress). **There are several interpretation of probability**: each of them analyse the concepts of probability from different point of view
- In this sense the probability of an event is a vague concept
- The standard definition of probability theory is a due to Kolmogorov's axiomatization
- NOTE: several of the leading "interpretations of probability" fail to satisfy all of Kolmogorov's axioms while other quantities that have nothing to do with probability can satisfy Kolmogorov's axioms [Hajek(2011)]

Kolmogorov's Probability Theory I

- Let Ω be a non-empty set i.e. the universal set. A field on Ω is a set F of subsets of Ω that has Ω as a member, and that is closed under complementation and union. Let P be a function from F to the real numbers obeying:
  * Non-negativity: $P(A) \geq 0$, for all $A \in F$.
  * Normalization: $P(\Omega) = 1$
  * Finite additivity: $P(A \cup B) = P(A) + P(B)$ for all $A, B \in F$ such that $A \cap B = \emptyset$

  then P is a probability function, and (Ω, F, P) a probability space
- A sigma field on Ω is a field F, closed under complementation and countable union, with the following property:
  * **Countable additivity**: if A_1, A_2, A_3 is a countably infinite sequence of disjoint sets, each of which is an element of F, then $P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$
- NOTE: this axiom allow the assimilation of probability theory to measure theory [Hajek(2011)]

Kolmogorov's Probability Theory II

- It is possible to attach probabilities to members of a collection S of sentences of a formal language, closed under (countable) truth-functional combinations, with the following axiomatization:
  * $P(A) \geq 0$ for all $A \in S$
  * If T is a logical truth (in classical logic), then $P(T) = 1$
  * $P(A \cup B) = P(A) + P(B)$ for all $A, B \in S$ such that A and B are logically incompatible
- The **conditional probability** of A given B is then given by the ratio of unconditional probabilities: $P(A|B) = \frac{P(A \cap B)}{P(B)}$, provided $P(B) > 0$

Probability Interpretations

- A part from the assignment of 1 to the universal set and 0 to the empty set, there are two main interpretations of probability:

* **objective (physical or frequency) probability**, is associated with random physical systems (roulette wheels and radioactive atoms). A given event tends to occur at "relative frequency", in a long run of trials. The two main kinds of this interpretation are **frequentist** and **propensity** [Hajek(2011)]
* **subjective (evidential or Bayesian) probability**, can be assigned to any statement, even when no random process is involved. It represent the subjective plausibility, or the degrees of belief to which the statement is supported by the available evidence. The four main kinds of this interpretation are the **classical, the subjective, the epistemic** (or inductive) and the **logical** [Hajek(2011)]

Frequentist Probability

- **The probability of an event A is the relative frequency of A over a large number of experiments**. The probability of an attribute A in a finite reference class B is the relative frequency of actual occurrences of A within B
- **Example**. Rolling a dice: if a player observes that over 6×10^6 experiments the number 2 occurred 10^6 times. Then, it can infer that the probability of the number 2 is $\frac{1}{6}$
- **Problem**: this interpretation is valid only if it is possible to repeat the experiment a huge (infinity) number of times and its probability never changes. Both these assumptions are not always (never) verified
- NOTE: frequentist interpretation remains the dominant view of probability in statistics, and in the sciences [Hajek(2010)]

Classical Probability

- **The key idea is the principle of indifference**: when there is no evidence or in the presence of symmetrically balanced evidence, the events have the same probability
- If a random experiment can result in N mutually exclusive and equally likely events and if N_A of these result in the occurrence of the event A, the probability of A is defined by $P(A) = \frac{N_A}{N}$
- **Example**: we toss a coin. Since there is no reason to believe that heads is more probable than tail or viceversa, it is possible to assume that they have the same probability
- **Problem**: it is applicable only to situations in which there is only a "finite" number of possible outcomes; the principle of indifference extracts information from ignorance
- NOTE: similar to frequentist probability: the crucial difference is that the frequentist interpretation counts actual events while the classical interpretation counts all the possible events of a given experiment [Hajek(2010)]

Bayesian Probability I

- **The probability of an event A is its degree of belief or confidence**
- The space of events E can be partitioned in term of the single events $\{A_i\}$ and its probability can be expressed by the law of total probability $P(E) = \sum_i P(A_i)P(E|A_i)$. The Bayes's theorem states that the conditional probability $P(A_j|E)$ can be written as

$$P(A_j|E) = \frac{P(A_j)P(E|A_j)}{P(E)} = \frac{P(A_j)P(E|A_j)}{\sum_i P(A_i)P(E|A_i)}$$

- NOTE: the Bayesian interpretation expresses how a subjective degree of belief should change to account for evidence while the frequentist interpretation relates inverse representations of the probabilities concerning two events
- NOTE: **subjective probabilities are traditionally analyzed in terms of betting behavior**. The standard approach is a due to de Finetti [Hajek(2011)]

Bayesian Probability II

- **The probability of an event** A is the amount of money α (betting-odd) that a rational and reversible bookmaker (BM) would propose for the following bet: a gambler (G) bets a real number λ and pays $\lambda\alpha$ to BM. If A will be true, then he will get back λ from BM, and if A will be false, then he will get nothing from him
- NOTE:
  - **In the actual betting games, the bookmaker declares the inverse $\frac{1}{\alpha}$ of α** that is, if the gambler pays λ and A will be true, then he will get $\frac{\lambda}{\alpha}$. This formulation is equivalent to the previous one (in both if the gambler pays $\lambda\alpha$ he will get $\frac{\lambda\alpha}{\alpha} = \lambda$) although it seems more inviting for the gambler
  - **the probability of an event A is not absolute, but subjective**. Different bookmaker may have different opinions about the betting odd α
  - **BM is a reversible bookmaker**. The role of the bookmaker and the gambler can be exchanged if the gambler bets a negative amount of money λ (paying $\lambda < 0$ is equal to receiving λ)

Bayesian Probability II

- **A book** Γ is a finite set of bets on different events $A_1, ..., A_n$ that BM accepts i.e. $\Gamma = (A_1, \alpha_1), ..., (A_n, \alpha_n)$, where for $i = 1, ..., n$, A_i is an event and α_i is BM's betting odd, that is, the amount of money that the BM chooses for a bet on A_i
- If for $i = 1, ..., n$, the gambler bets λ_i (where the λ_i may be negative) on A_i, and the truth value of A_i is $v(A_i)$, then BM's payoff will be

$$\sum_{i=1}^{n} \lambda_i(\alpha_i - v(A_i))$$

- **A rational bookmaker**, according to de Finetti, is a bookmaker with a book $\Gamma = (A_1, \alpha_1), ..., (A_n, \alpha_n)$, called rational or coherent, that does not contains a winning strategy or Dutch Book for the gambler i.e. there is no system of bets $\lambda_1, ..., \lambda_n$ on $A_1, ..., A_n$ such that the gambler's payoff

$$\sum_{i=1}^{n} \lambda_i(v(A_i) - \alpha_i) > 0 \equiv \sum_{i=1}^{n} \lambda_i(\alpha_i - v(A_i)) < 0$$

or independently of the truth values $v(A_1), ..., v(A_n)$

Bayesian Probability III

Theorem 1

A book $\Gamma = (A_1, \alpha_1), ..., (A_n, \alpha_n)$ is rational or coherent iff there is a probability distribution P such that $P(A_i) = \alpha_i$, $i = 1, ..., n$. Therefore, a book is rational if it respects the laws of probability:

- the probability of an event A is a number between 0 and 1
- the probability of the certain event is 1
- if two events A, B are incompatible (i.e. they never occur simultaneously) the probability of their union is the sum of their probabilities i.e. $P(A \cup B) = P(A) + P(B)$

Bayesian Probability IV

- **Example**. Two teams A and B are going to play a soccer game. Then book A wins $\frac{2}{5}$, B wins $\frac{2}{5}$, Draw $\frac{2}{5}$ is not coherent: if the gambler bets -1 on all events, then the bookmaker will pay $\frac{6}{5}$ and get back 1 whatever the result is, with a sure loss of $\frac{1}{5}$
- **Problem**. The event A: on next horse race, I will win a million of euros. If the bookmaker chooses the betting-odd 1 for A, then the gambler has no winning strategy. From the de Finetti's pont of view it is coherent to assume that the event A has probability 1
- NOTE: although the de Finetti's approach is valid from a theoretical point of view, sometimes does not match the reality
- NOTE: the de Finetti's point of view is that the belief of a rational agent should be respected even if it look strange: e.g. in the previous example the agent could know for sure that the race is not fair

Fuzzy Logic and Probability Theory

- **Fuzzy Logic and Probability Theory are different ways of expressing uncertainty**: Zimmermann calls them linguistic and stochastic uncertainty, respectively
- **Fuzzy Logic is a logic of partial degree of truth and it refers to vague propositions** i.e. fuzzy set theory uses the concept of fuzzy set membership to express degree of truth
- **Probability Theory is a theory of the degree of belief of the truth value of the propositions and it refers to crisp propositions** i.e. Probability Theory uses the concept of subjective probability (partial knowledge) to trying to make predictions about events
- Although the distinction is mostly philosophical, the Fuzzy Logic measure is different from the Probability Theory measure so they yield different models of the same real-world situations i.e. they are mathematically similar but conceptually distinct

Relationship between Fuzzy Logic and Probability

- Mathematically, the main difference is that **the degrees of belief are not truth-functional** (extensional) i.e. the probability of $p \wedge q$ is not a function of probability of p and q while **degrees of truth of vague propositions are truth-functional**
- **Fuzzy Logic is a Many-Valued Logic**, whereas **Probability Theory is based on Classical Logic**
- "Fuzzy Logic is not a poor man's probability theory" [Hajek(1995)]
- "The numerous schemes for representing and reasoning about uncertainty that have appeared in the AI literature are unnecessary - probability is all that is needed" [Cheesman]

Probability Theory inside Fuzzy Logic

- It is possible to treat **Probability Theory inside Fuzzy Logic**. The key idea is to define the probability of an event A as the truth value of the sentence: Probably, A i.e. $Pr(A)$
- The Lukasiewicz's Logic allow to define a sort of sum $A \oplus B = \neg(\neg A \&_T \neg B) = min\{1, A + B\}$. Hence the additive property of a probability can be expressed by the rule if $A \rightarrow_T \neg B$ then $Pr(A \cup B) \leftrightarrow Pr(A) \oplus Pr(B)$
- This allows to treat the coherence of an assessment in probability as logical coherence of a finite set of formulas of the logic
- NOTE: in this way is possible to built an elegant bridge between fuzziness and probability, with a simple axiom system over Lukasiewicz's Logic [Hajek(2010)]
- NOTE: if two formulas are Lukasiewicz-equivalent then they are classically equivalent, but the converse does not hold

Actual point of view

- **Many statisticians**, from the de Finetti's school, **think that only one kind of mathematical uncertainty is needed** and thus Fuzzy Logic is unnecessary
- **Kosko** maintains that **Probability Theory is a subtheory of Fuzzy Logic**, since probability only handles one kind of uncertainty
- **Zadeh argues that Fuzzy Logic is different in nature from Probability Theory**, and is not a replacement for it. He "fuzzified" probability to fuzzy probability and generalized it to a new theory called Perception-based Probability Theory [Zadeh(2002)]

Possible extensions

- In literature there are several approaches that try to create a **probabilistic reasoning in logical framework**. Although almost all logics of probability are based on Classical Logic, there are some exceptions:
  * the Fuzzy Probability Theory of Hajek [Hajek(1995)]
  * the Perception-based Probability Theory of Zadeh [Zadeh(2002)]
- NOTE: nowadays the relationship between the Fuzzy Logic and the Probability Theory is still an open research problem

Fuzzy Probability fuzzy theory I

- The *Fuzzy Probability* (FP) Theory is described in the framework of the truth-functional RPL which is naturally related to probability theory
- **Lukasiewicz's Logic only allows to prove tautologies while RPL admits graded formulas and graded proof** i.e. inference from partially true assumptions to conclusions which can be partially true
- The key idea of FP is that probability maintains classical equivalence therefore understands formulas as crisp propositions, but probability is just a variable
- Given a crisp formula ϕ is possible to define a new propositional variable f_ϕ such that the probability $Pr(\phi)$ is equal to the truth degree of f_ϕ, read as "ϕ is probable"

Fuzzy Probability fuzzy theory II

- **A generalization of classical Godel's completeness theorem is provable in RPL** [Gottwald(2009)] and consequently in FP
- Hajek shows that is possible for FP to have an axiomatization probabilistically complete [Hajek(1995)]
- Further, FP is able to describe other uncertainty models different from probability like Possibility Theory where formulas are valued with possibility and necessity degrees
- To each crisp formula ϕ is associate a fuzzy formula f_ϕ read as "ϕ is necessary" or "ϕ is certain":
  * the truth-degree of f_ϕ represents the necessity degree of ϕ
  * the truth-degree of $\neg f_{\neg\phi}$ represents the possibility degree of ϕ

Perception-based Probability Theory

- Since **in the world almost everything is a matter of degree**, Zadeh states that there is fundamental conflict between bivalence and reality
  * in *Probability Theory* (PT), which is based on Classical Logic, only likelihood is a matter of degree i.e. the **logic of measurements**
  * in *Perception-based Probability Theory* (PTp), which is based on Fuzzy Logic, everything is, or can be, a matter of degree i.e. **the logic of perceptions**
- The **PTp** proposed by Zadeh is the **computational theory of perceptions**, where perceptions are treated through their descriptions in a natural language
  * perceptions are intrinsically imprecise and this imprecision is a due to limited ability of the sensors (also natural sensors like brain) to resolve detail and store information
- The **Zadeh's key idea in PTp is that subjective probabilities are perception of belief** (likelihood) and as such are intrinsically imprecise

From Probability Theory to Perception-based Probability Theory I

- PTp is the result of a three-stage generalization of PT:
  1. **f-generalization** leads to PT^+, a probability theory in which probabilities, events and relations are, or are allowed to be, fuzzy-set-valued
     * e.g. the probability of an event may be described as "very high" and a relation between X and Y may be defined as "X is much larger than Y"
  2. **f.g-generalization** leads to a probability theory PT^{++} in which probabilities, events and relations are, or are allowed to be, f-granular
     * e.g. if $Y = f(X)$, then f may be described as a collection of fuzzy IF-THEN rules of the form: if X is A_i then Y is B_i, $i = 1, \ldots, n$, in which the A_i and B_i are fuzzy sets in the domains of X and Y, respectively
  3. **nl-generalization** leads to PTp which has the capability to operate on perception-based information. A key concept is the *Precisiated Natural Language* (PNL) that consists of those propositions in a *Natural Language* (NL) which are precisiable through translation into a precisiation language called *Generalized Constraint Language* (GCL). A proposition p in NL may be represented in PNL as a generalized constraint of the form $X \text{ is} r\ R$ where:
     * X is the constrained variable
     * R is the constraining relation
     * r is an indexing variable that defines the way in which R constrains X

     The principal types of constraints are possibilistic (r=blank); probabilistic (r=p); veristic (r=v); and usuality (r=u). Thus, $X \text{ is} u\ R$ means that usually (X is R)

From Probability Theory to Perception-based Probability Theory II

- NOTE: in general, X, R and r are implicit in p. Thus, in PNL, representation of the meaning of p requires explicitation of X, R and r
  * e.g the meaning of "Eva is young", can be represented as "Age(Eva) is young", where $X = Age(Eva)$, $R = young$, and the constraint is possibilistic in the sense that "young" defines the fuzzy set of possible values of Age(Eva)
- PNL plays an essential role in:
  1. representation of imprecise probabilities and probability distributions
  2. definition of basic concepts such as independence and stationarity
  3. deduction from perception-based information
- NOTE: in PTp the default assumption is that probability distributions are bimodal and this lead to a expected value that, in general, fuzzy-set-valued
- NOTE: **PTp is more complex than PT** but this is the price price of constructing a probability theory which has a close rapport with the imprecision, uncertainty and ill-definedness of the real world [Zadeh(2002)]

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